

同步练习 P219 = .1

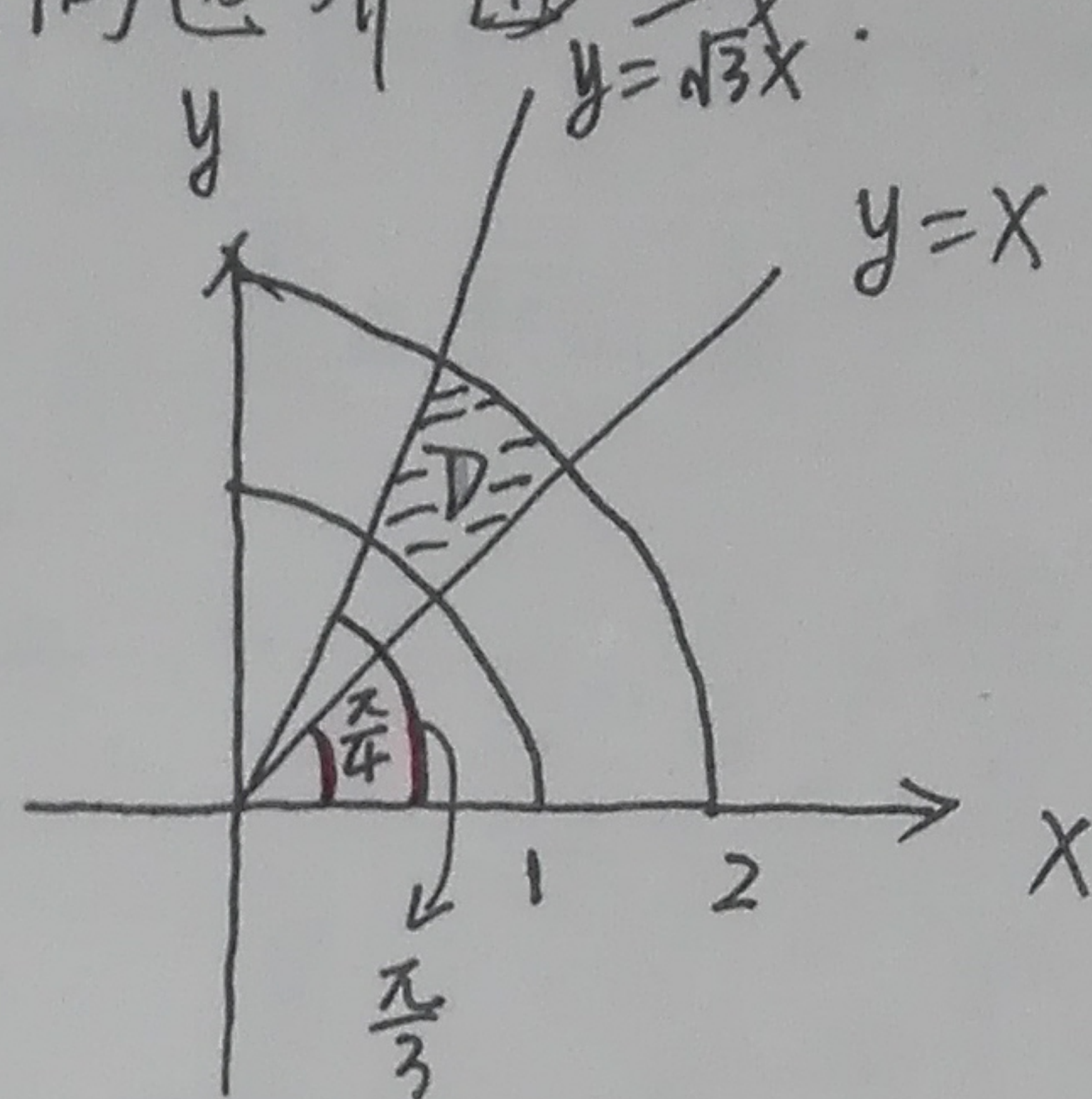
廖国勇 2016.4.18

1. 求 $\oint_C \frac{1}{x} \arctan \frac{y}{x} + \frac{2}{y} \arctan \frac{x}{y} dy$, 式中 C 是曲线 $x^2+y^2=1$, $x^2+y^2=4$, $y=x$, $y=\sqrt{3}x$ 在第 I 象限围成区域 D 的正向边界曲线.

解: D 区域由极坐标表示为:

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{3} \quad 1 \leq \rho \leq 2$$

$$P = \frac{1}{x} \arctan \frac{y}{x} \quad Q = \frac{2}{y} \arctan \frac{x}{y}$$



$$\frac{\partial P}{\partial y} = \frac{1}{x} \cdot \frac{\frac{1}{x}}{1 + (\frac{y}{x})^2} = \frac{1}{x^2 + y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{2}{y} \cdot \frac{\frac{1}{y}}{1 + (\frac{x}{y})^2} = \frac{2}{x^2 + y^2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{x^2 + y^2}$$

$$\text{于是, } \oint_C \frac{1}{x} \arctan \frac{y}{x} + \frac{1}{y} \arctan \frac{x}{y} dy = \oint_C P dx + Q dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_D \frac{1}{x^2 + y^2} dx dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta \int_1^2 \frac{1}{\rho^2} \cdot \rho d\rho$$

$$= \ln 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta$$

$$= \frac{\pi}{12} \ln 2$$

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2. 求 $\oint_C \frac{ydx - xdy}{4(x^2 + y^2)}$, 式中 C 是圆 $(x-1)^2 + y^2 = 2$, 取逆时针方向.

分析: C 围成的区域包含 $(0,0)$, $P = \frac{y}{4(x^2 + y^2)}$, $Q = \frac{-x}{4(x^2 + y^2)}$ 在 $(0,0)$ 处没有定义, 更不用说有连续的偏导数, 故挖掉点 $(0,0)$

解: 设 C_1 是圆 $x^2 + y^2 = r^2$, $0 < r < \sqrt{2} - 1$, 取逆时针方向

C 与 C_1 围成的区域为 D , D 的正方向边界为 $C + C_1^-$

$$P = \frac{y}{4(x^2 + y^2)} \quad Q = \frac{-x}{4(x^2 + y^2)}$$

$$\frac{\partial P}{\partial y} = \frac{4(x^2 + y^2) - 8y^2}{4^2(x^2 + y^2)^2} = \frac{4x^2 - 4y^2}{4^2(x^2 + y^2)^2} = \frac{x^2 - y^2}{4(x^2 + y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{-4(x^2 + y^2) + 8x^2}{4^2(x^2 + y^2)^2} = \frac{4x^2 - 4y^2}{4^2(x^2 + y^2)^2} = \frac{x^2 - y^2}{4(x^2 + y^2)^2}$$

$$\text{于是, } \int_{C+C_1^-} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = 0$$

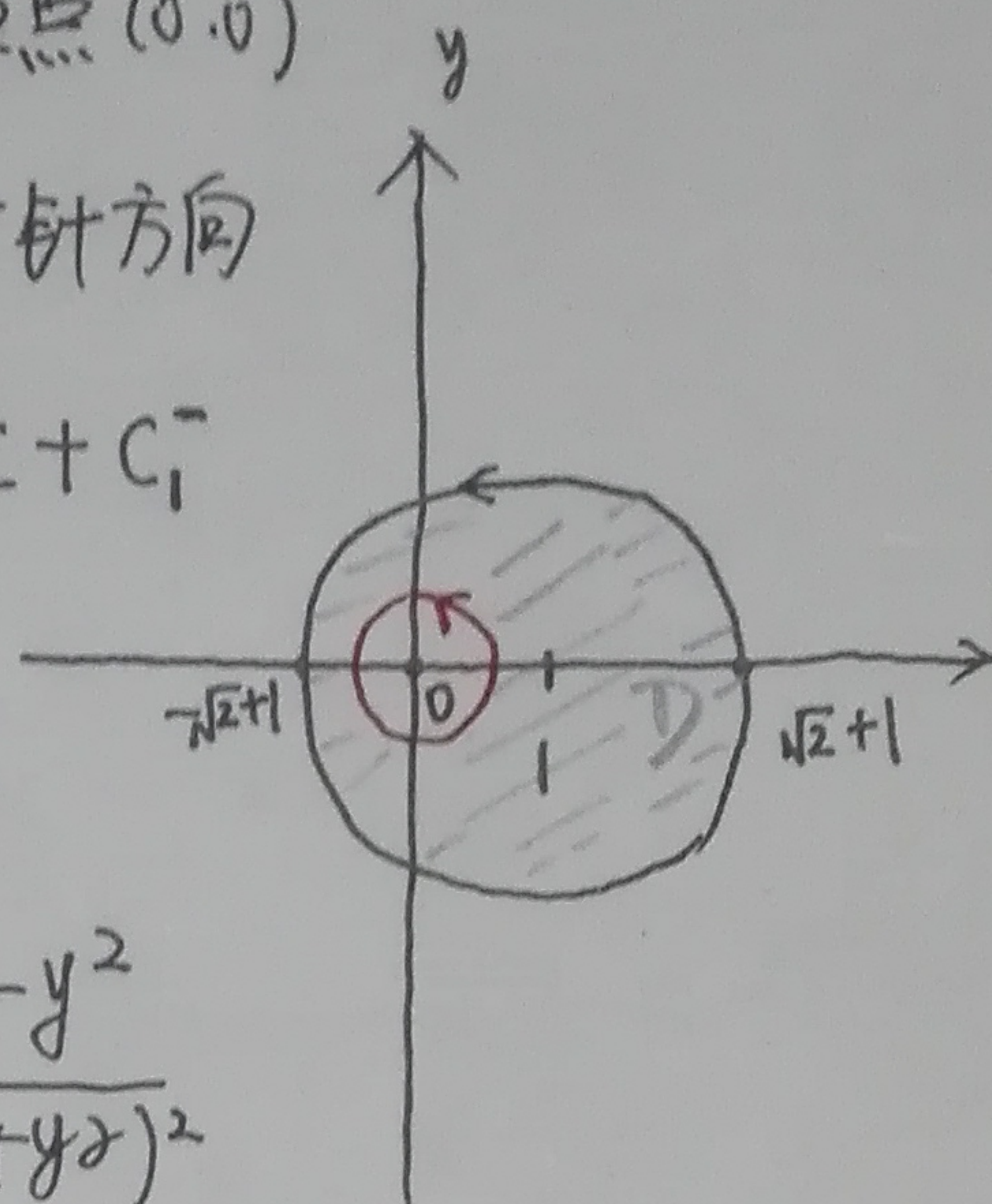
$$\text{即 } \oint_C Pdx + Qdy = \oint_{C_1} Pdx + Qdy$$

$$\oint_C \frac{ydx - xdy}{4(x^2 + y^2)} = \oint_{C_1} \frac{ydx - xdy}{4(x^2 + y^2)}$$

$$= \int_0^{2\pi} \frac{r \sin \theta d(r \cos \theta) - r \cos \theta d(r \sin \theta)}{4r^2}$$

$$= \int_0^{2\pi} -\frac{1}{4} d\theta$$

$$= -\frac{\pi}{2}$$



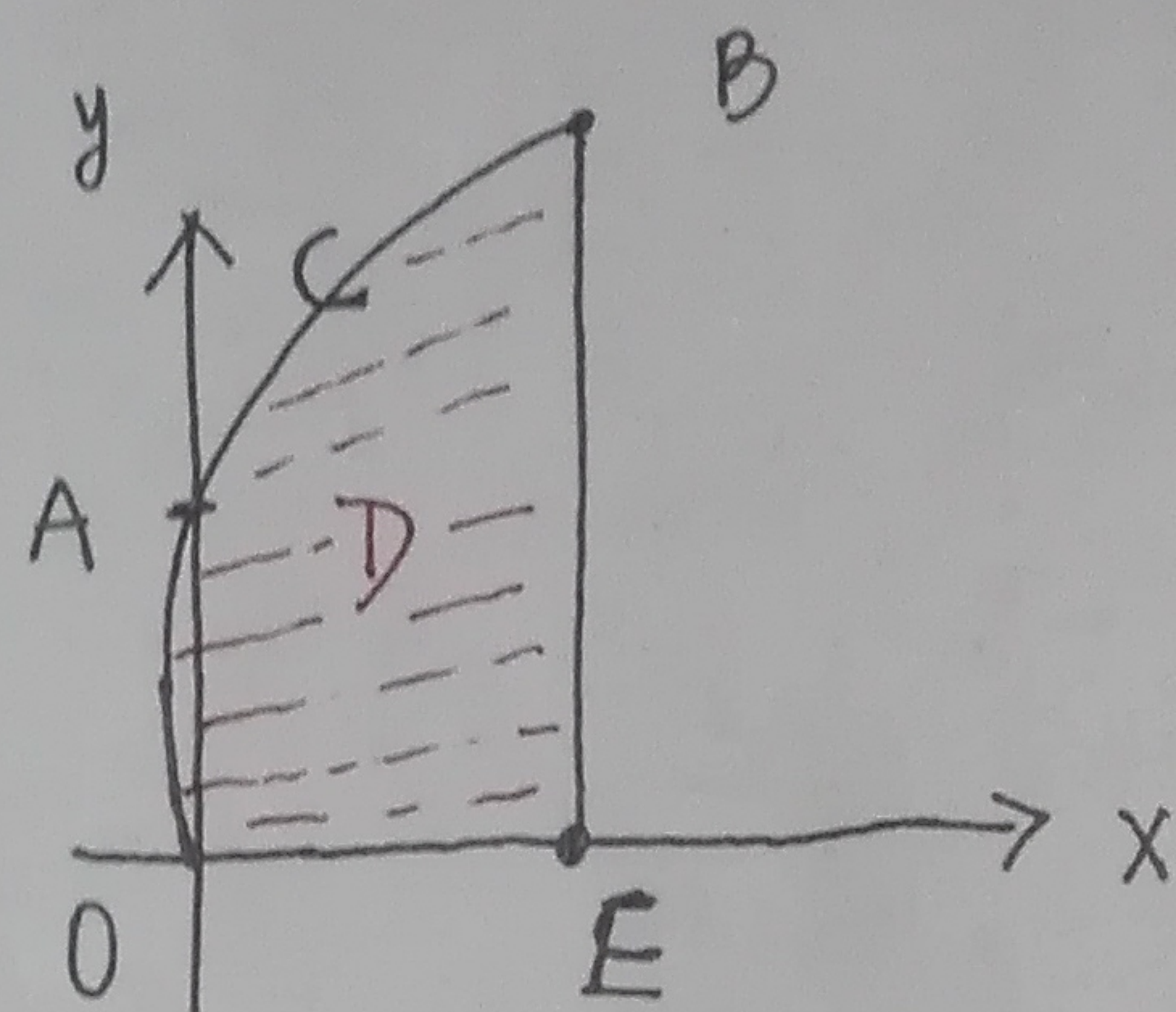
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3. 求 $\oint_C (e^y + 3x^2) dx + (xe^y + 2y) dy$, 式中 C 是过 $O(0,0)$, $A(0,1)$, $B(1,2)$ 的圆弧, 从点 O 到点 B .

解: 在 x 轴上取一点 $E(1,0)$

则曲线 C 与 BE , EO 围成一区域, 设为 D

显然 $C+BE+EO$ 为 D 的负方向.



$$p = e^y + 3x^2, \quad Q = xe^y + 2y \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^y - e^y = 0$$

$$\text{于是, } \int_{C+BE+EO} p dx + Q dy = - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = 0$$

$$\text{即 } \oint_C p dx + Q dy = - \int_{BE} p dx + Q dy - \int_{EO} p dx + Q dy$$

$$\int_{BE} p dx + Q dy = \int_2^0 (e^y + 3 \cdot 1^2) d1 + (1 \cdot e^y + 2y) dy$$

$$= \int_2^0 (e^y + 2y) dy = (e^y + y^2) \Big|_2^0 = 1 - e^2 - 4$$

$$\int_{EO} p dx + Q dy = \int_1^0 (e^0 + 3x^2) dx + (x \cdot e^0 + 2 \cdot 0) d0$$

$$= \int_1^0 (1 + 3x^2) dx = (x + x^3) \Big|_1^0 = -2$$

$$\text{所以 } \oint_C (e^y + 3x^2) dx + (xe^y + 2y) dy = \oint_C p dx + Q dy$$

$$= -(1 - e^2 - 4) - (-2) = 5 + e^2$$

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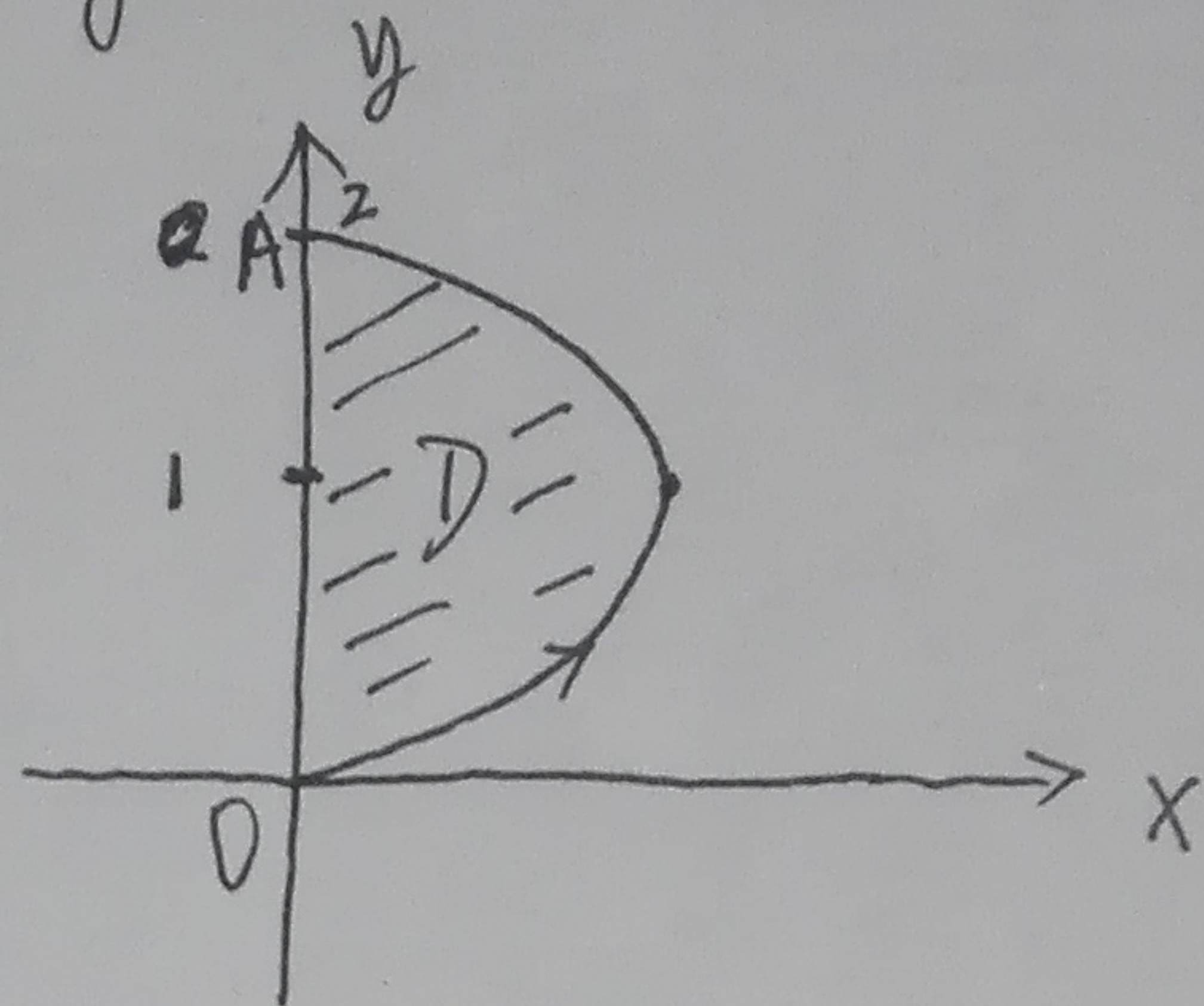
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4. 求 $\int_C (x^2 - e^x \cos y) dx + (e^x \sin y + 3x) dy$, 式中 C 是圆 $x = \sqrt{2y - y^2}$ 从 $O(0,0)$ 到 $A(0,2)$ 的一段弧.

解 C 与直线段 AO 围成一区域,

设该区域为 D , 显然 $C+AO$

为 D 的正向边界.



$$P = x^2 - e^x \cos y \quad \frac{\partial P}{\partial y} = e^x \sin y$$

$$Q = e^x \sin y + 3x \quad \frac{\partial Q}{\partial x} = e^x \sin y + 3$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (e^x \sin y + 3) - e^x \sin y = +3$$

$$\text{于是, } \int_{C+AO} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_D +3 dx dy = +\frac{3}{2}\pi$$

$$\int_C P dx + Q dy = +\frac{3}{2}\pi - \int_{AO} P dx + Q dy$$

$$= +\frac{3}{2}\pi + \int_{OA} P dx + Q dy$$

$$= +\frac{3}{2}\pi + \int_0^2 (0^2 - e^0 \cos y) dy + (e^0 \sin y - 3 \cdot 0) dy$$

$$= +\frac{3}{2}\pi + \int_0^2 \sin y dy = -\frac{3}{2}\pi + 1 - \cos 2$$

$$= \frac{3}{2}\pi + 1 - \cos 2$$

$$\text{即 } \int_C (x^2 - e^x \cos y) dx + (e^x \sin y + 3x) dy = \frac{3}{2}\pi + 1 - \cos 2$$

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5. 求 $\oint_C (x+y)^2 dx + (x^2-y^2) dy$, 式中 C 是 $A(1,1), B(3,2), D(3,5)$ 为顶点的三角形正向边界曲线.

解: AB 直线方程为: $y = \frac{1}{2}x + \frac{1}{2}$

AD 直线方程为: $y = 2x - 1$

区域 D : $1 \leq x \leq 3, \frac{1}{2}x + \frac{1}{2} \leq y \leq 2x - 1$

$$P = (x+y)^2, Q = x^2 - y^2$$

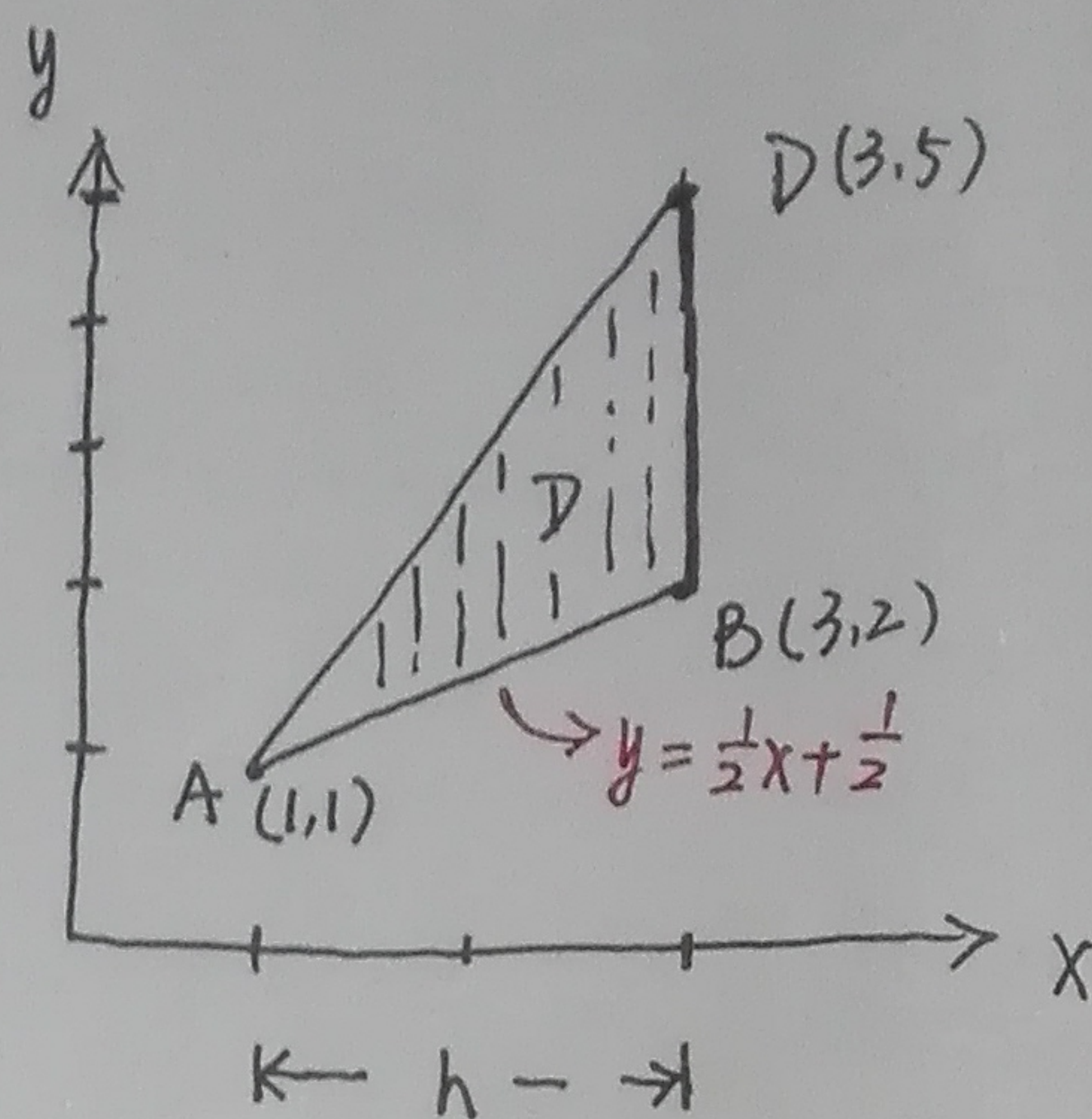
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2(x+y) = -2y$$

$$\text{于是, 原式} = \oint_C P dx + Q dy = \iint_D -2y dy dx$$

$$= \int_1^3 dx \int_{\frac{1}{2}x + \frac{1}{2}}^{2x-1} -2y dy$$

$$= \int_1^3 \left[\left(\frac{1}{2}x + \frac{1}{2} \right)^2 - (2x-1)^2 \right] dx$$

$$= -16$$



注, $\iint_D -2y dx dy = -2\bar{y} S_D$ $\bar{y}$ 为三角形 ABD 的中心点的纵坐标

$$\bar{y} = \frac{1}{3}(1+2+5) = \frac{8}{3} \quad S_D = \frac{1}{2} \cdot \text{底} \cdot \text{高} = \frac{1}{2} BD \cdot h = 3$$

$$\text{于是 } \iint_D -2y dx dy = -2 \cdot \frac{8}{3} \cdot 3 = -16$$

此仍巧算, 不懂就按常规计算.

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6. 求 $\int_C (x^2 + 2xy)dx + (x^2 + y^4)dy$, 式中 C 是曲线 $y = \sin \frac{\pi}{2}x$ 从点 $O(0,0)$ 到点 $A(2,0)$ 的一段弧.

解: $C+AO$ 围成一区域, 设为 D

显然 $C+AO$ 是 D 边界的负方向.

$$P = x^2 + 2xy \quad Q = x^2 + y^4$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2x = 0$$

$$\int_{C+AO} Pdx + Qdy = - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0$$

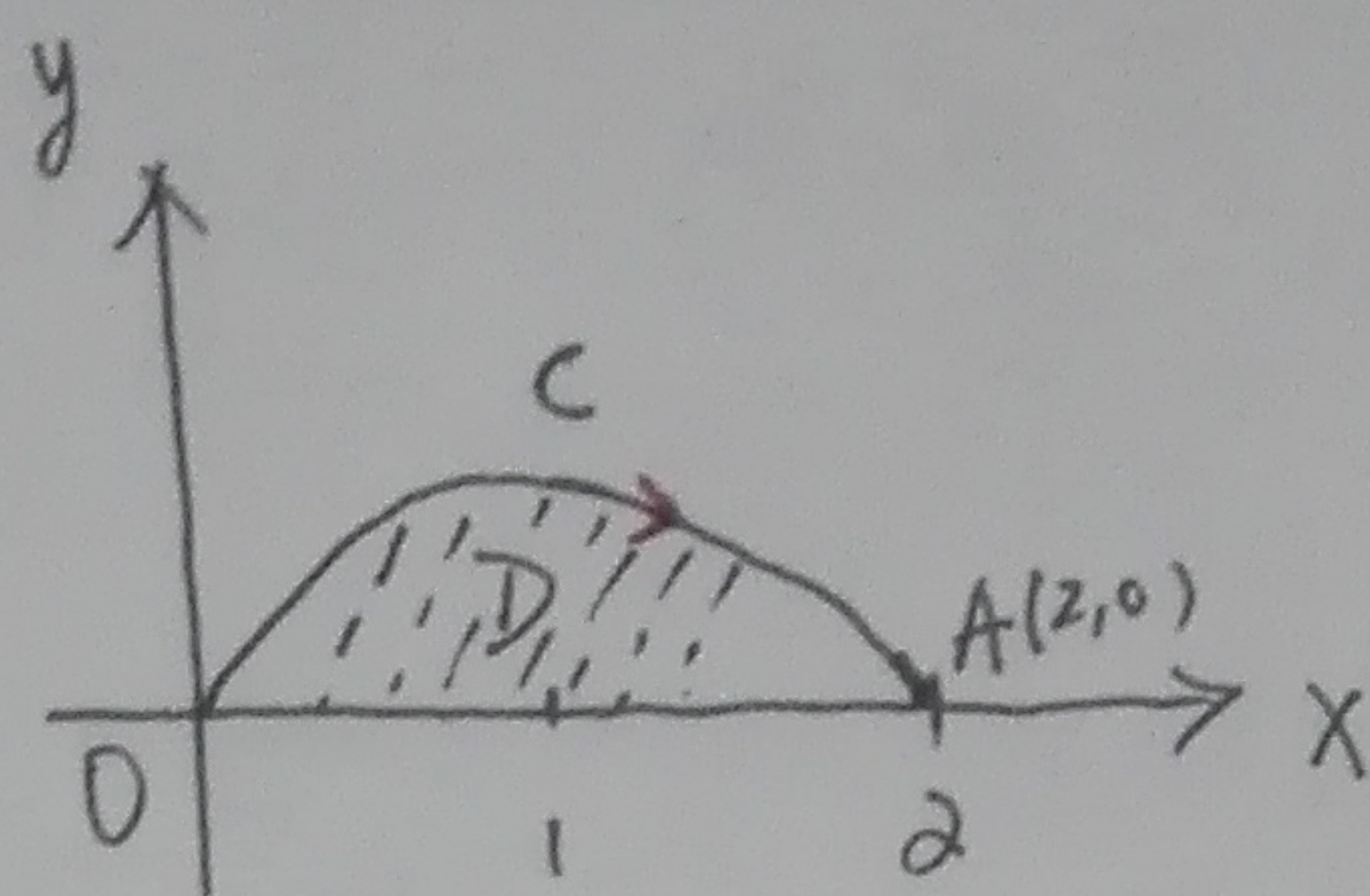
$$\int_C Pdx + Qdy = - \int_{AO} Pdx + Qdy$$

$$= \int_{OA} Pdx + Qdy$$

$$= \int_0^2 (x^2 + 2x \cdot 0) dx + (x^2 + 0^4) d0$$

$$= \int_0^2 x^2 dx = \frac{8}{3}$$

$$\text{原式} = \frac{8}{3}$$



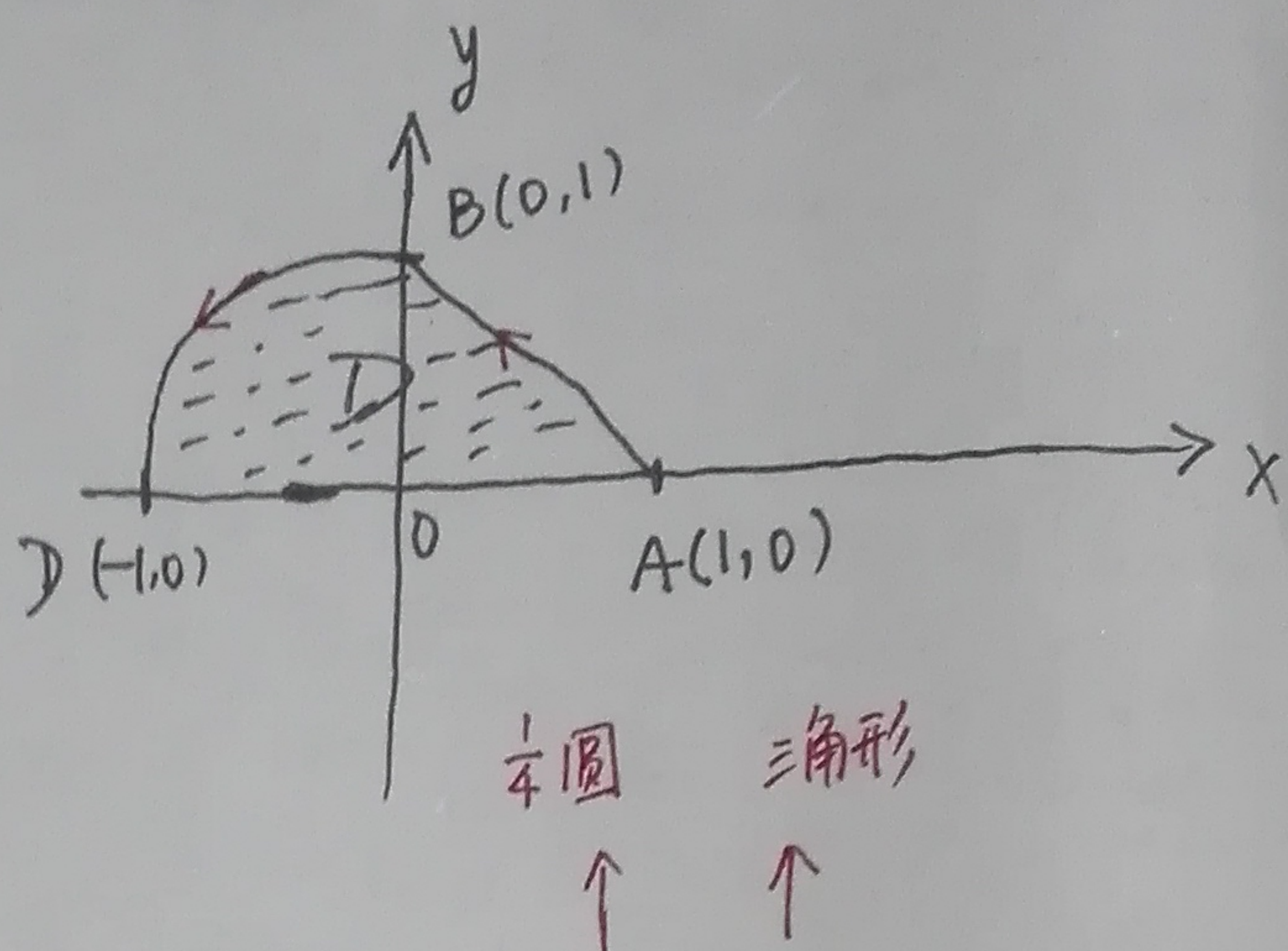
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7. 求 $\int_C (x + e^{\sin y}) dy - (y - \frac{1}{2}) dx$, 式中 C 为从点 $A(1, 0)$ 沿 $x+y=1$ 到点 $B(0, 1)$, 再沿 $x^2+y^2=1$ 到点 $D(-1, 0)$ 的曲线段.

解 $C+DA$ 围成区域 D

显然 $C+DA$ 是 D 的边界的方向

$$Q = x + e^{\sin y} \quad P = -(y - \frac{1}{2})$$



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

$$\int_{C+DA} P dx + Q dy = \iint_D 2 dx dy = 2 \int_D = 2(\frac{\pi}{4} + \frac{1}{2}) = \frac{\pi}{2} + 1$$

$$\int_C P dx + Q dy = \frac{\pi}{2} + 1 - \int_{DA} P dx + Q dy$$

$$= \frac{\pi}{2} + 1 - \int_{-1}^1 (x + e^{\sin 0}) d\sigma - (0 - \frac{1}{2}) dx$$

$$= \frac{\pi}{2} + 1 - \int_{-1}^1 \frac{1}{2} dx$$

$$= \frac{\pi}{2}$$

所以, 原式 = $\frac{\pi}{2}$

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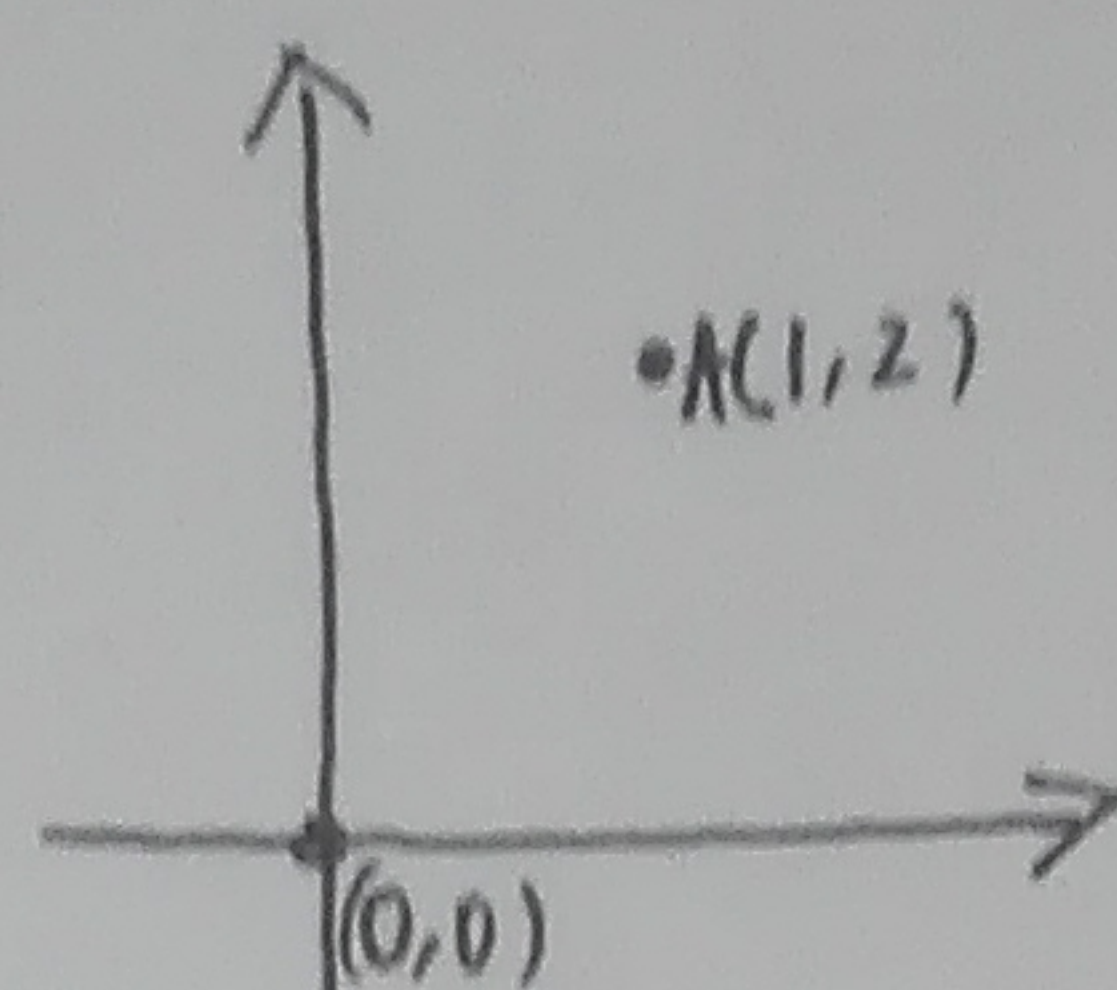
8. 求 $\int_{(0,0)}^{(1,2)} (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy$

解: $P = x^4 + 4xy^3$. $Q = 6x^2y^2 - 5y^4$ $\frac{\partial P}{\partial y} = 12xy^2 = \frac{\partial Q}{\partial x}$

故积分与路径无关。

方法一，选一条简单的路径。

选直线段 OA: $y = 2x$ $0 \leq x \leq 1$



于是，原式 $= \int_0^1 [x^4 + 4x \cdot (2x)^3] dx + [6x^2 \cdot (2x)^2 - 5 \cdot (2x)^4] d(2x)$

$$= \int_0^1 -79x^4 dx = -\frac{79}{5}$$

方法二，全微分法

$$u(x,y) = \int P dx = \int (x^4 + 4xy^3) dx = \frac{x^5}{5} + 2x^2y^3 + \varphi(y)$$

$$\frac{\partial u(x,y)}{\partial y} = 6x^2y^2 + \varphi'(y) = Q(x) = 6x^2y^2 - 5y^4$$

$$\varphi'(y) = -5y^4$$

$\varphi(y)$ 取 $-y^5$ 即可

于是 $u(x,y) = \frac{x^5}{5} + 2x^2y^3 - y^5$

$$\int_{(0,0)}^{(1,2)} (x^4 + 4xy^3) dx + (6x^2y^2 - 5y^4) dy$$

$$= u(x,y) \Big|_{(0,0)}^{(1,2)} = u(1,2) - u(0,0)$$

$$= \frac{1}{5} + 2 \cdot 1^2 \cdot 2^3 - 2^5 = -\frac{79}{5}$$

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求函数 $u(x, y)$, 使得 $du = (2x \cos y + y^2 \cos x) dx + (2y \sin x - x^2 \sin y) dy$

解: $u = \int (2x \cos y + y^2 \cos x) dx$

$$= x^2 \cos y + y^2 \sin x + \phi(y)$$

$$\frac{\partial u}{\partial y} = x^2 (-\sin y) + 2y \sin x + \phi'(y)$$

$$= 2y \sin x - x^2 \sin y$$

$$\Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = C$$

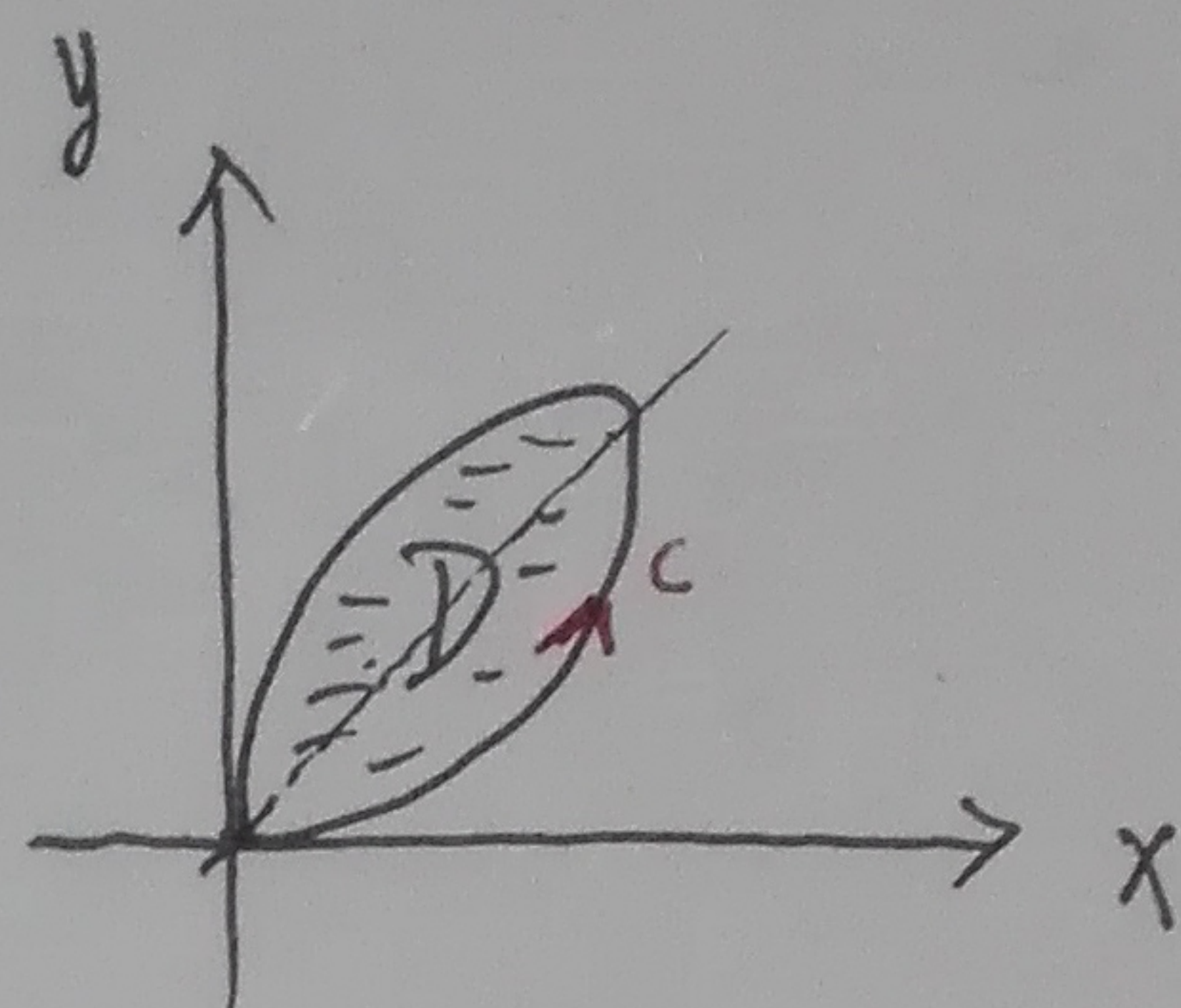
于是 $u(x, y) = x^2 \cos y + y^2 \sin x + C$

一般情况下 C 取 0.

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求 $\oint_C (x+y) dy - dx$, 式中 C 是曲线 $\rho = \sin 2\theta$ 上从点 θ 到点 $\theta = \frac{\pi}{2}$ 的弧段.

解: C 围成一区域 D , 且 C 为区域 D 的边界正方向.



$$\oint_C (x+y) dy - dx$$

$$= \iint_D \left[\frac{\partial}{\partial x} (x+y) - \frac{\partial}{\partial y} (1) \right] dx dy$$

$$\text{这里 } P = -1, Q = x+y$$

$$= \iint_D dx dy$$

$$D: \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\sin 2\theta} \rho d\rho$$

$$0 \leq \rho \leq \sin 2\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2 2\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \cos^2 2\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \cdot \frac{1}{2} (\sin^2 2\theta + \cos^2 2\theta) d\theta$$

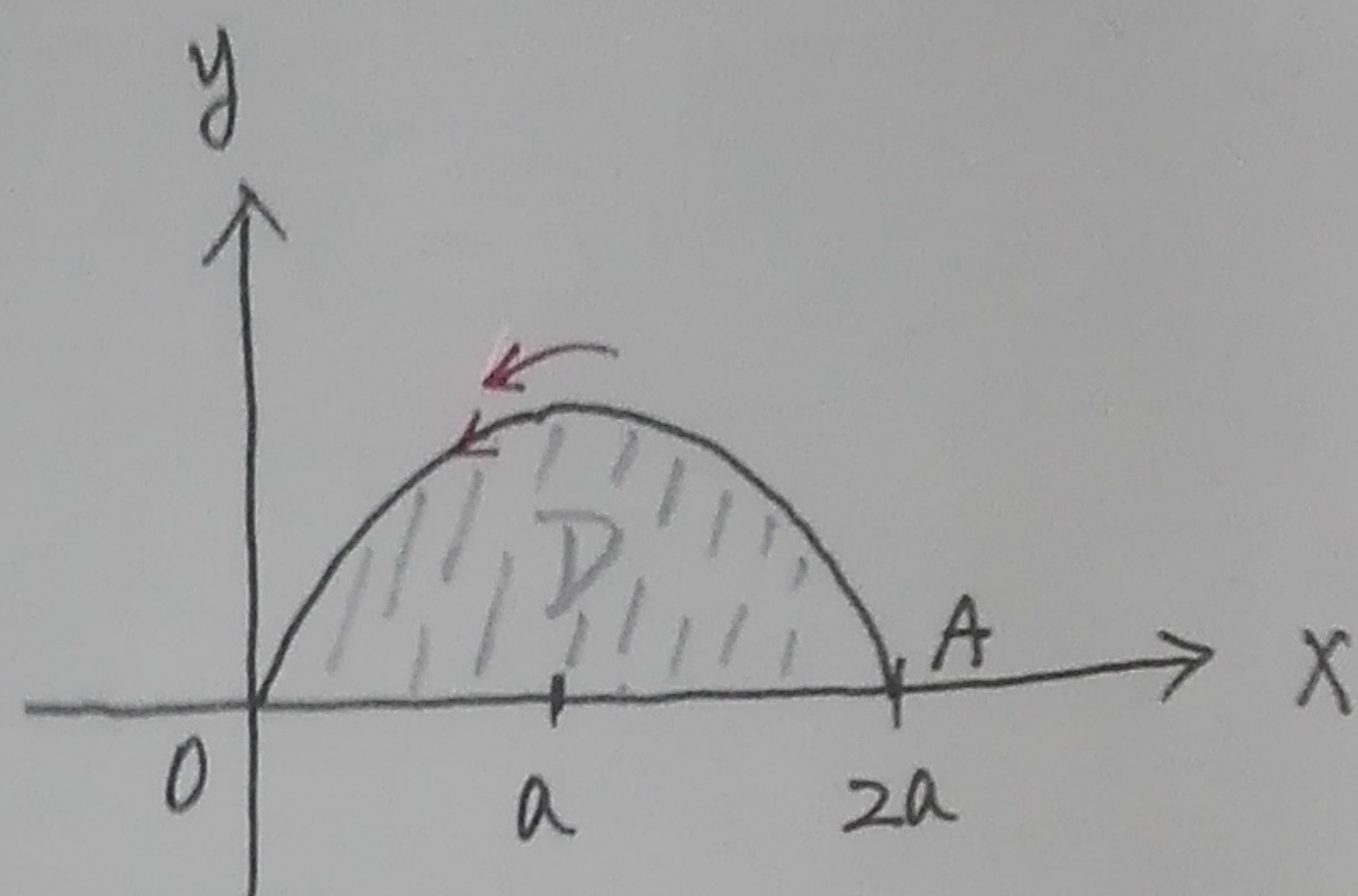
$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} d\theta = \frac{\pi}{8}$$

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11. 求 $\int_C [e^x \sin y - b(x+y)] dx + (e^x \cos y - ax) dy$, 式中 a, b 为常数, C 是圆 $y = \sqrt{2ax - x^2}$ 上从点 $A(2a, 0)$ 到点 $O(0, 0)$ 的一段弧.

解: $C + OA$ 构成一区域 D 的边界正方向.



$$P = e^x \sin y - b(x+y)$$

$$Q = e^x \cos y - ax$$

$$\frac{\partial Q}{\partial x} = (e^x \cos y - a) \quad \frac{\partial P}{\partial y} = e^x \cos y - b \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = b - a$$

$$\text{原式} = \int_{C+OA} - \int_{OA}$$

$$= \iint_D (b-a) dx dy - \int_0^{2a} [e^x \sin 0 - b(x+0)] dx + (e^x \cos 0 - ax) d0$$

$$= (b-a) S_D + \int_0^{2a} bx dx$$

$$= (b-a) \frac{\pi}{2} a^2 + 2a^2 b$$

$$= \frac{\pi}{2} a^2 b - \frac{\pi}{2} a^3 + 2a^2 b$$